

# Flat 3-webs via semi-simple Frobenius 3-folds

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## Abstract

We construct flat 3-webs via semi-simple geometric Frobenius manifolds of dimension three and give geometric interpretation of the Chern connection of the web. These webs turned out to be biholomorphic to the characteristic webs on the solutions of the corresponding associativity equation. We show that such webs are hexagonal and admit at least one infinitesimal symmetry at each singular point. Singularities of the web are also discussed.

**Key words:** hexagonal 3-web, Frobenius manifold, Chern connection.

**AMS Subject classification:** 53A60 (primary), 53D45 (secondary).

## 1 Introduction

B.Dubrovin introduced the notion of Frobenius manifold geometrizing the physical context of WDVV-equations that arise originally in the two-dimensional topological field theory (see [7] and [8]). Frobenius manifold is a complex analytic manifold  $M$  equipped with the following analytic objects:

- commutative and associative multiplication on  $T_p M$ ,
- invariant non-degenerate flat inner product :  $\langle u \cdot v, w \rangle = \langle u, v \cdot w \rangle$ ,
- constant unity vector field  $e$ :  $\nabla e = 0$ ,  $e \cdot v = v \ \forall v \in TM$ ,
- linear Euler vector field  $E$ :  $\nabla(\nabla E) = 0$ ,

satisfying the following conditions:

1. one-parameter group generated by  $E$  re-scales the multiplication and the inner product,
2. 4-tensor  $(\nabla_z c)(u, v, w)$  is symmetric in  $u, v, w, z$ , where

$$c(u, v, w) := \langle u \cdot v, w \rangle.$$

In the above definition the symbol  $\nabla$  stands for the Levi-Civita connection of the inner product  $\langle \cdot, \cdot \rangle$ .

Consider a semi-simple Frobenius manifold  $M$  of dimension 3, which means that the algebra  $T_p M$  is semi-simple for each  $p \in U$  for some open set  $U \subset M$ . Then  $T_p M$  is a direct product of one-dimensional algebras spanned by idempotents:

$$T_p M = \mathbb{C}\{e_1\} \otimes \mathbb{C}\{e_2\} \otimes \mathbb{C}\{e_3\}, \quad e_i \cdot e_j = \delta_{ij}.$$

In this setting the unity vector field is

$$e = e_1 + e_2 + e_3.$$

Let  $S$  be a surface transverse to the unit vector field  $e$ , then 3 planes spanned by  $\{e, e_i\}$  cut 3 directions on  $T_p S$  (see Fig.1 on the left). Integral curves of these direction fields build a **flat or hexagonal 3-web**, i.e. this web is locally biholomorphic to 3 families of parallel lines. Further we will see

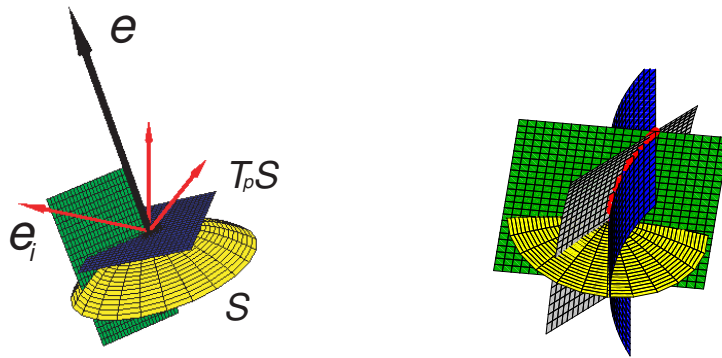


Figure 1: Construction of a booklet 3-webs from Frobenius 3-folds.

that the distributions  $\{e, e_i\}$  are integrable, integral surfaces of each of the distributions being formed by the integral curves of the unity vector field  $e$ . Thus for any point in  $M$  there are 3 integral surfaces intersecting along a such curve. These surfaces cut  $S$  along the constructed web. This justifies the following definition (see also Fig.1 on the right).

**Definition 1** *The constructed web is called a booklet 3-web.*

We call a point **regular or non-singular** if the web directions are pairwise transverse. Hexagonal 3-web does not have any local invariants in regular points. Its "personality" is encoded in the behavior at **singular points**, where at least 2 web directions coincide.

For the booklet 3-webs constructed above we show that:

- the booklet 3-web is biholomorphic to the characteristic 3-web of the corresponding solution of associativity equation,
- the booklet 3-web has at least one infinitesimal symmetry at each singular point,
- the Chern connection of the booklet 3-web is induced by the connection on  $TM$  compatible with the algebraic structure of  $M$ .

Finally, we discuss possible singularities of the constructed webs.

## 2 Characteristic Webs

Consider in more details the local construction described in the introduction. Suppose the operator of multiplication by the Euler vector field

$$\mu_E : T_p M \rightarrow T_p M, \quad v \mapsto E \cdot v$$

has pairwise distinct eigenvalues  $\lambda_1(p), \lambda_2(p), \lambda_3(p)$ . Then the functions  $\lambda_i(p)$  are local coordinates called also **canonical** or **Dubrovin** coordinates. In these coordinates  $\partial_i \cdot \partial_j = \delta_{ij} \partial_i$ , where  $\partial_i = \partial/\partial \lambda_i$  (see [6]). Therefore the basis vectors  $\partial_i$  are idempotents  $\partial_i = e_i$ . Moreover, holds true  $E = \sum_i \lambda_i \partial_i$ . The distributions spanned by the pairs  $\{e_i, e\}$ , where  $e$  is the unity  $e = e_1 + e_2 + e_3$ , are determined by the following 1-forms:

$$\theta_1 = d\lambda_2 - d\lambda_3, \quad \theta_2 = d\lambda_3 - d\lambda_1, \quad \theta_3 = d\lambda_1 - d\lambda_2. \quad (1)$$

It is immediate that  $\sum_i \theta_i = 0$  and  $d\theta_i = 0$ . Let  $\iota : U \subset \mathbb{C}^2 \rightarrow S \subset M$  be a local parametrization of a surface  $S$  transverse to the unity vector field  $e$ . Define  $\omega_i := \iota^*(\theta_i)$ , then

$$\sum_i \omega_i = \sum_i \iota^*(\theta_i) = \iota^* \left( \sum_i \theta_i \right) = 0, \quad d\omega_i = d(\iota^*(\theta_i)) = \iota^*(d(\theta_i)) = 0.$$

Thus the following Theorem is immediate.

**Theorem 1** *The booklet 3-web is hexagonal.*

*Proof:* Each of the distributions  $\omega_i = 0$  on  $S$  defines locally a first integral  $u_i$  by  $du_i = \omega_i$ . These integrals sum up to a constant:  $u_1 + u_2 + u_3 = \text{const}$ , which implies hexagonality of the web. (Take  $u_1$  and  $u_2$  as local coordinates on  $S$ .)  $\square$

Suppose the Lie derivative of the unity vector field  $e$  does not vanish:  $\mathcal{L}_E(e) \neq 0$ . Dubrovin [7] showed that any Frobenius structure with this property is locally defined by a potential  $F(t)$ ,  $t = (t^1, \dots, t^n)$  such that its 3d order derivatives  $c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$  satisfy the following conditions.

1. Normalization: the matrix  $\eta_{\alpha\beta} = c_{1\alpha\beta}(t)$  is constant and not degenerate. This matrix gives inner product  $\langle \partial_\alpha, \partial_\beta \rangle = \eta_{\alpha\beta}$ , the parameters  $(t^1, \dots, t^n)$  being its flat coordinates.
2. Associativity: the functions  $c_{\alpha\beta}^\gamma := \sum_i \eta^{\gamma i} c_{i\alpha\beta}(t)$  define a structure of associative algebra on  $T_p M$ :

$$\partial_{t^\alpha} \cdot \partial_{t^\beta} = \sum_i c_{\alpha\beta}^i \partial_{t^i},$$

with  $(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}$ .

3. Homogeneity:  $F(c^{w_1} t^1, \dots, c^{w_n} t^n) = c^{w_F} F(t^1, \dots, t^n)$ .

The algebra is automatically commutative, its unity is  $e = \partial_1$ , the Euler vector field is given by  $E = \sum_i w_i t^i \partial_i$ . Associativity condition manifests itself as a system of nonlinear PDEs, called associativity or Witten-Dijkgraaf-Verlinde-Verlinde equations.

For Frobenius 3-folds there is only one associativity equation of order three. There are two non-equivalent normal forms.

**Example 1** If  $\langle e, e \rangle = 0$  then in some flat coordinates the matrix  $\eta$  is anti-diagonal

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the potential has the form

$$F(t) = \frac{1}{2}t^2y + \frac{1}{2}tx^2 + f(x, y), \text{ where } t^1 = t, t^2 = x, t^3 = y.$$

Associativity equation reads as

$$f_{yyy} = f_{xxy}^2 - f_{xxx}f_{xyy}.$$

Recall that characteristic curves  $\varphi(x_1, x_2) = \text{const}$  for a solution of a PDE  $H(x_1, x_2, f, f_1, \dots, f_J) = 0$ , where  $J = (j_1, j_2)$  with  $j_1 + j_2 \leq 3$  is a multi-index and  $f_J = \frac{\partial^{|J|}f}{\partial x_1^{j_1} \partial x_2^{j_2}}$ , are defined by the following implicit PDE

$$\sum_{|J|=3} \frac{\partial H}{\partial f_J} \nabla \varphi^J = 0, \quad \nabla \varphi^J := \varphi_1^{j_1} \varphi_2^{j_2}. \quad (2)$$

For the above equation this PDE takes the form

$$\varphi_y^3 + f_{xxx}\varphi_y^2\varphi_x - 2f_{xxy}\varphi_y\varphi_x^2 + f_{xyy}\varphi_x^3 = 0$$

As the associativity equation is non-linear the web of characteristics (we will call it also a **characteristic 3-web**) depends on the solution. Solving this equation is equivalent to integration of an implicit cubic ODE. Substituting  $[\varphi_y : \varphi_x] = [dx : -dy]$  one arrives at the following cubic **binary form**:

$$dx^3 - f_{xxx}dx^2dy - 2f_{xxy}dxdy^2 - f_{xyy}dy^3 = 0. \quad (3)$$

The multiplication table for the corresponding Frobenius algebra is

$$\begin{aligned} \partial_x \cdot \partial_x &= f_{xxy}\partial_t + f_{xxx}\partial_x + \partial_y, \\ \partial_x \cdot \partial_y &= f_{xyy}\partial_t + f_{xxy}\partial_x, \\ \partial_y \cdot \partial_y &= f_{yyy}\partial_t + f_{xyy}\partial_x. \end{aligned}$$

**Example 2** If  $\langle e, e \rangle \neq 0$  then in some flat coordinates the matrix  $\eta$  is

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and the potential has the form

$$F(t) = \frac{1}{6}t^3 + txy + f(x, y), \text{ where } t^1 = t, t^2 = x, t^3 = y.$$

Associativity equation reads as

$$f_{xxx}f_{yyy} - f_{xxy}f_{xyy} = 1.$$

The characteristic 3-web in  $(x, y)$ -plane is defined by the PDE

$$f_{xxx}\varphi_y^3 - f_{xxy}\varphi_y^2\varphi_x - f_{xyy}\varphi_y\varphi_x^2 + f_{yyy}\varphi_x^3 = 0$$

or by the cubic binary form:

$$f_{xxx}dx^3 + f_{xxy}dx^2dy - f_{xyy}dxdy^2 - f_{yyy}dy^3 = 0. \quad (4)$$

The multiplication table is

$$\begin{aligned} \partial_x \cdot \partial_x &= f_{xxy}\partial_x + f_{xxx}\partial_y, \\ \partial_x \cdot \partial_y &= \partial_t + f_{xyy}\partial_x + f_{xxy}\partial_y, \\ \partial_y \cdot \partial_y &= f_{yyy}\partial_x + f_{xyy}\partial_y. \end{aligned}$$

**Remark 1.** Note that the coefficients of the binary forms (3) and (4) never vanish simultaneously. For this reason, the web directions are well-defined everywhere. They can coincide thus giving multiple roots of the binary equations.

**Remark 2.** The above two associativity equations are equivalent with respect to some non-local coordinate transform, depending on solutions (see [3]). This transform preserves the hexagonality of the web.

It was observed (see [9]) that the characteristic 3-webs are hexagonal. Now we show that they are diffeomorphic to the above constructed booklet webs.

**Theorem 2** *The booklet 3-web is biholomorphic to the characteristic 3-web of the corresponding solution of associativity equation.*

*Proof:* Let us check this claim for the plane  $t = \text{const}$ . Set  $N = T\partial_t + X\partial_x + Y\partial_y$  for an idempotent. The coefficients by  $\partial_t, \partial_x, \partial_y$  of the vector equation  $N \cdot N - N = 0$  give 3 scalar equations. The coefficient by  $\partial_y$  gives  $T$ . Substituting this expression into the coefficient by  $\partial_x$  one gets the binary differential forms for  $[dx : dy] = [X, Y]$  that coincide with (3) or (4) respectively. For the general case choose a plane  $P = \{(t, x, y) : t = \text{const}\}$  and consider the flow generated by the unity vector field. This flow preserves the distributions  $\theta_i = 0$ . For each point  $p \in S$  of the surface the orbit of the point cuts the plane  $P$  at a point  $\psi(p) \in P$ , thus giving a desired biholomorphism  $\psi$ .  $\square$

### 3 Infinitesimal Symmetries

We say that a 3-web on a surface with local coordinates  $x, y$  has an infinitesimal symmetry

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y,$$

if the local flow of the vector field  $X$  maps the web leaves to the web leaves. Infinitesimal symmetries form a Lie algebra with respect to the Lie bracket. Cartan proved (see [5]) that at a regular point a 3-web either does not have infinitesimal symmetries (generic case), or has a one-dimensional symmetry algebra (then in suitable coordinates it can be defined by the form

$$dx \cdot dy \cdot (dy + g(x + y)dx) = 0, \quad g \neq \text{const}$$

with the symmetry  $\partial_y - \partial_x$ ), or has a three-dimensional symmetry algebra (then it is equivalent to the web defined by the form

$$dx \cdot dy \cdot (dy + dx) = 0$$

with the symmetry algebra generated by  $\{\partial_x, \partial_y, x\partial_x + y\partial_y\}$ ). In the last case, when the symmetry algebra has the largest possible dimension 3, the 3-web is hexagonal. Note that not all symmetries survive at a singular point. The condition to have at least one-dimensional symmetry at a singular point is not trivial. The following binary equation

$$dy^3 - 2x^2y(1 + x^2)dydx^2 + 8x^3y^2dx^3 = 0$$

defines a flat 3-web but loses all its symmetries at the singular point  $(0, 0)$ .

**Theorem 3** *The booklet 3-web has at least one infinitesimal symmetry at each point.*

*Proof:* The flow  $\exp(aE)$  of the Euler vector field  $E$  respects the distributions  $\theta_i = 0$  and induces an action  $T_a$  on the surface  $S$ . This action preserves the web. Indeed, let  $p \in S$  be a point of the surface and  $C_p$  the orbit of  $p$  under the flow of the unity vector field  $e$ . Then the action  $T_a$  maps  $p$  to the point of intersection of  $S$  with the image of  $C_p$  under  $\exp(aE)$ , i.e.  $T_a(p) := \exp(aE)C_p \cap S$ .  $\square$

### 4 Chern Connection

For computing Chern connection of 3-webs we use Blaschke's approach based on differential forms [4]. Let us introduce **binary vector field** as a section of

the vector bundle with the base manifold  $U \in \mathbb{C}^2$  and the fiber of symmetric forms  $V$  on  $T_p^*U$ , i.e.  $V$  is a  $n$ -linear symmetric map  $(T_p^*U)^n \rightarrow \mathbb{C}$ . Then equation (2) has the form  $V(d\varphi) = 0$ , where

$$V = a(x, y)(\partial_x)^3 + b(x, y)(\partial_x)^2\partial_y + c(x, y)\partial_x(\partial_y)^2 + r(x, y)(\partial_y)^3 \quad (5)$$

is a cubic binary vector field. (Do not confuse monomials in  $\partial_x, \partial_y$  with the product in Frobenius algebra, defined in introduction!)

At a non-singular point the field  $V$  can be factorized:

$$V = V_1 V_2 V_3, \quad \text{where } V_i = q_i(x, y)\partial_x - p_i(x, y)\partial_y, \quad i = 1, 2, 3 \quad (6)$$

and

$$\sigma_i = p_i dx + q_i dy, \quad i = 1, 2, 3,$$

are the roots of  $V$ , which means  $V(\sigma_i) = 0$ . Due to the homogeneity of  $V$  these roots are defined up to the multiplication by a non-vanishing factor. They can be normalized to satisfy the condition

$$\sigma_1 + \sigma_2 + \sigma_3 = 0. \quad (7)$$

Upon introducing an "area" form by

$$\Omega = \sigma_1 \wedge \sigma_2 = \sigma_2 \wedge \sigma_3 = \sigma_3 \wedge \sigma_1 = (p_1 q_2 - p_2 q_1) dx \wedge dy,$$

define the Chern connection form as

$$\gamma := h_2 \sigma_1 - h_1 \sigma_2 = h_3 \sigma_2 - h_2 \sigma_3 = h_1 \sigma_3 - h_3 \sigma_1,$$

where  $h_i$  are determined by

$$d\sigma_i = h_i \Omega.$$

The web is flat iff the connection form is closed:  $d(\gamma) = 0$ . This implies  $d\sigma_i = \gamma \wedge \sigma_i$ . Putting

$$dk = -\gamma k, \quad (8)$$

we introduce first integrals  $u_i$  of the foliations (at least locally) at a regular point by

$$du_1 = k\sigma_1, \quad du_2 = k\sigma_2, \quad du_3 = k\sigma_3, \quad (9)$$

satisfying the equation  $u_1 + u_2 + u_3 = 0$ , which implies hexagonality.

**Remark.** Let  $\eta_1, \eta_2, \eta_3$  be germs of differential forms in  $(\mathbb{C}^2, q_0)$  defining a flat 3 web and satisfying the following conditions:

- the forms are closed:  $d(\eta_i) = 0, \quad i = 1, 2, 3,$



- the forms define the web:  $\eta_i \wedge \sigma_i = 0$ ,  $i = 1, 2, 3$ ,
- the forms sum up to zero:  $\eta_1 + \eta_2 + \eta_3 = 0$ ,

then these forms are proportional to  $k\sigma_i$ :  $\eta_i = ck\sigma_i$ ,  $i = 1, 2, 3$ ,  $c = \text{const}$ . One says that the space of abelian relations is one-dimensional for a hexagonal 3-web. In other words the first integrals summing up to zero are defined up to a constant factor.

**Proposition 1** *In the normalization (7) the 3-web defined by the binary 3-vector (5) has the Chern connection*

$$\gamma = \frac{\gamma_1 dx + \gamma_2 dy}{3D},$$

where

$$D = 18abc r - 27a^2 r^2 - 4ac^3 + b^2 c^2 - 4b^3 r$$

is the discriminant of the equation  $V(\sigma) = 0$  and

$$\begin{aligned} \gamma_1 &= (15bcr - 27ar^2 - 4c^3) a_x + 6r(3br - c^2) a_y \\ &\quad + 2b(c^2 - 3br) b_x + 3r(bc - 9ar) b_y \\ &\quad + b(9ar - bc) c_x + 6r(3ac - b^2) c_y \\ &\quad + 2b(b^2 - 3ac) r_x + (9abr - 12ac^2 + 3b^2 c) r_y \\ \gamma_2 &= (9acr - 12b^2 r + 3bc^2) a_x + 2c(c^2 - 3br) a_y + \\ &\quad 6a(3br - c^2) b_x + c(9ar - bc) b_y + \\ &\quad 3a(9ar - bc) c_x + 2c(b^2 - 3ac) c_y \\ &\quad 6a(3ac - b^2) r_x + (15abc - 27a^2 r - 4b^3) r_y. \end{aligned}$$

*Proof:* With  $p_3 = -p_1 - p_2$ ,  $q_3 = -q_1 - q_2$  equation (6) is equivalent to the following 4 equations:

$$\begin{aligned} q_1 q_2 (q_1 + q_2) &= -a, \quad p_1 p_2 (p_1 + p_2) = r, \\ p_1 (q_2^2 + 2q_1 q_2) + p_2 (q_1^2 + 2q_1 q_2) &= b, \\ q_1 (p_2^2 + 2p_1 p_2) + q_2 (p_1^2 + 2p_1 p_2) &= -c. \end{aligned} \tag{10}$$

Differentiating each of these equations with respect to  $x$  and  $y$  one gets all the derivatives of  $p_i, q_i$ . Substitution of these expressions into the formula for  $\gamma$  and simplification with the help of equations (10) yields the resulting formula for  $\gamma$ .  $\square$

**Corollary 1** *In the normalization (7) the characteristic 3-web of associativity equations is*

$$\gamma = -\frac{1}{6}d(\ln D), \quad (11)$$

where the components  $a, b, c, r$  of the binary vector are given by equation (2).

Using the form  $\gamma$  Blaschke (see [4]) introduced a parallel transport of the tangent vectors. The construction is the following: take vector fields  $v_i$ ,  $i = 1, 2, 3$  tangent to the web curves, normalize them to satisfy  $v_1 + v_2 + v_3 = 0$ , and choose, for example,  $\{v_1, v_2\}$  as a frame on the tangent bundle. Then each vector field  $\xi$  can be represented locally as  $\xi = \xi^1 v_1 + \xi^2 v_2$  and the corresponding connection is defined by the following forms on the tangent bundle:

$$\Gamma^1 = d\xi^1 - \gamma\xi^1, \quad \Gamma^2 = d\xi^2 - \gamma\xi^2,$$

i.e. the vector field  $\xi$  is parallel along a curve iff the forms  $\Gamma^i$  vanish along it. In particular, if our 3-web is hexagonal then the vectors  $v_i$  can be normalized to commute. In this normalization holds true  $\gamma = 0$  and the vector fields with  $\xi^i = \text{const}$  are parallel along any curve.

Now we give a geometric interpretation of the Chern connection of the characteristic web. At a regular point the idempotents  $\{e_1, e_2, e_3\}$  form a frame of the tangent bundle  $TM$ :  $\eta = \eta^1 e_1 + \eta^2 e_2 + \eta^3 e_3$  for each vector field  $\eta$  on  $M$ . The projections of two of them, say, of  $e_1, e_2$  along the unity  $e$  form a frame  $\{v_1, v_2\}$  on  $S$ .

**Theorem 4** *Suppose  $v = \eta^1 e_1 + \eta^2 e_2 + \eta^3 e_3 \in T_p S$  and a curve  $\alpha : I \mapsto S$  with  $\alpha(I) \in S$ ,  $\alpha(0) = p$  does not pass through singular points of the booklet 3-web. Consider the vector field  $\eta \in TM$  along  $\alpha$  such that the coordinates  $\eta^i$  are kept constant. The projection of  $\eta$  into  $T_{\alpha(t)} S$  along  $e$  is the parallel transport defined by the Chern connection.*

*Proof:* It is sufficient to check that the vector fields  $v_1, v_2 \in TS$  commute. Holds true  $\theta_1(e_1) = 0$ ,  $\theta_2(e_1) = -1$ , where the forms  $\theta_i$  are defined by (1). Setting  $e_1 = v_1 + k_1 e$  one obtains  $\theta_1(e_1) = \theta_1(v_1) = \omega_1(v_1) = 0$  and  $\theta_2(e_1) = \theta_2(v_1) = \omega_2(v_1) = -1$ . Similar  $\omega_1(v_2) = 1$  and  $\omega_2(v_2) = 0$ . The forms  $\omega_i$  are closed therefore the vector fields  $v_1, v_2$  commute.  $\square$

**Remark.** The connection on  $TM$  defined by Theorem 4 preserves the idempotents and therefore respects the algebraic structure of the Frobenius manifold.

## 5 Singularities of Characteristic 3-webs

In [2] hexagonal 3-webs were studied via implicit cubic ODEs, obtained from a binary differential form

$$K_3(x, y)dy^3 + K_2(x, y)dy^2dx + K_1(x, y)dydx^2 + K_0(x, y)dx^3 = 0, \quad (12)$$

where the coefficients  $K_i$  do not vanish simultaneously, by deviding by  $dx^3$  or  $dy^3$ . Namely, if the equation is brought to the form without the quadratic term

$$p^3 + A(x, y)p + B(x, y) = 0,$$

then its Chern connection is

$$\gamma = \frac{(2A^2Ax - 4A^2By + 6ABAx + 9BBx)}{4A^3 + 27B^2}dx + \frac{(4A^2Ay + 6ABx + 18BBx - 9BAx)}{4A^3 + 27B^2}dy. \quad (13)$$

Note that, in general, the connection form becomes singular on the discriminant curve of the web. A direct calculation shows that the condition (11) is equivalent to the fact that the connection (13) remains holomorphic in the singular points. Therefore the germ of the connection form is exact:  $\gamma = df$ , where  $f$  is some function germ.

**Theorem 5** [1] *Suppose ODE (12) admits an infinitesimal symmetry  $X$  vanishing at the point  $(0, 0)$  on the discriminant curve  $\Delta$  and the germ of the Chern connection form is exact  $\gamma = d(f)$ , where  $f$  is some function germ. Then the equation germ and the symmetry are biholomorphic to one of the following normal forms:*

- |                                                                                            |                                            |
|--------------------------------------------------------------------------------------------|--------------------------------------------|
| 1) $y^{m_0}p^3 - p = 0,$                                                                   | $X = (2 + m_0)x\partial_x + 2y\partial_y,$ |
| 2) $p^3 + 2xp + y = 0,$                                                                    | $X = 2x\partial_x + 3y\partial_y,$         |
| 3) $(p - \frac{2}{3}x)(p^2 + \frac{2}{3}xp + y - \frac{2}{9}x^2) = 0,$                     | $X = x\partial_x + 2y\partial_y,$          |
| 4) $p^3 + 4x(y - \frac{4}{9}x^3)p + y^2 + \frac{64}{81}x^6 - \frac{32}{9}yx^3 = 0,$        | $X = x\partial_x + 3y\partial_y,$          |
| 5) $p^3 + y^2p = \frac{2}{\sqrt{27}}y^3 \tan(2\sqrt{3}x),$                                 | $X = y\partial_y,$                         |
| 6) $p^3 + y^{3+m_0}p = y^{\frac{9+3m_0}{2}}F\left([(m_0 + 1)]xy^{\frac{1+m_0}{2}}\right),$ | $X = (1 + m_0)x\partial_x - 2y\partial_y,$ |

where  $m_0$  is a non-negative integer and  $F(t)$  solves

$$[12 + 2t^2 - 9tF]\frac{dF}{dt} = \frac{2(m_0 + 3)}{m_0 + 1}(4 + 27F^2)$$

with  $F(0) = 0$ . The weights  $[w_1 : w_2]$  uniquely determine the normal form.

Characteristic 3-webs of associativity equations have an infinitesimal symmetry at each singular point and holomorphic Chern connections in the normalization (13), therefore their singularities are equivalent to the forms in Theorem 5. Somewhere else we will address the question if each of the above singularities type realizes via characteristic 3-webs of associativity equations.

## 6 Concluding Remarks

### 6.1 Generalization

The geometric construction, presented in this paper, can be generalized to higher dimensions. As a result we obtain, for  $n$ -dimensional Frobenius manifold, a collection of  $n$  commuting vector fields  $v_i$  in  $(\mathbb{C}^{n-1}, 0)$  satisfying the equation  $\sum_{i=1}^n v_i = 0$ , i.e., a flat  $n$ -web germ of curves in  $(\mathbb{C}^{n-1}, 0)$  admitting a "linear" symmetry.

### 6.2 Acknowledgement

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